Relative Disturbance Gain Array

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Disturbance rejection capabilities of different controller structures, for example, diagonal, block diagonal or full multivariable controller, are discussed. A generalized version of Relative Disturbance Gain, Generalized Relative Disturbance Gain (GRDG), is defined to evaluate the disturbance rejection capabilities of all possible controller structures. Furthermore, the relative disturbance gain array (RDGA) is introduced. Basic properties of RDGA are derived. An important one is: GRDG of all possible controller structures can be calculated directly from the array. Therefore, with RDGA, the synthesis of the controller structure can be done in a straightforward manner. Physical implications and quantitative analyses of GRDG are given. These form the basis for the synthesis. Finally, frequency-dependent GRDG is developed which evaluates the performance further based on dynamic information. Several examples are used to illustrate the synthesis of the controller structure. The results show that better disturbance rejection can be achieved by selecting appropriate controller structure.

Introduction

Disturbance rejection is the major objective in chemical process control. Without disturbances, a process will be kept at steady state and control is not necessary. When a disturbance comes into a multivariable system, all the outputs will be impacted. Several approaches such as two degree of freedom controllers (Shunta and Luyben, 1972; Stephanopoulos and Huang, 1986) and parallel cascade control (Yu, 1988; Shen and Yu, 1991) were proposed to improve the disturbance rejection capabilities. These control strategies are complicated owing to the fact that two sets of controllers are employed. Regardless of their complexity, these approaches assume types of controller structure *a priori*. By controller structure, we mean the controller takes the form of multiloop SISO controllers, a full multivariable controller or any other possible forms, as shown in Figure 1. Before going into such levels of complexity, one question should be asked is that: can we achieve better disturbance rejection by manipulating the controller structure, for example, will multiloop SISO controllers give better disturbance rejection than a full multivariable controller. Niederlinski (1971) is among the first to investigate the relationship between disturbance rejection capability and controller structures. The work of Niederlinski (1971) reveals that multiloop SISO controllers can give better load rejection than the inverse-based multivariable controller for a distillation column. Notice that this is a specific result, not a general conclusion about inverse-based controllers in distillation control. However, the analyses of Niederlinski (1971) require the process transfer function matrix as well as the controller parameters. This limits the applicability of Niederlinski approach. Based on the process and load transfer functions, Stanley et al. (1985) proposed the relative disturbance gain (RDG) which is defined as a ratio of the manipulated variable under perfect

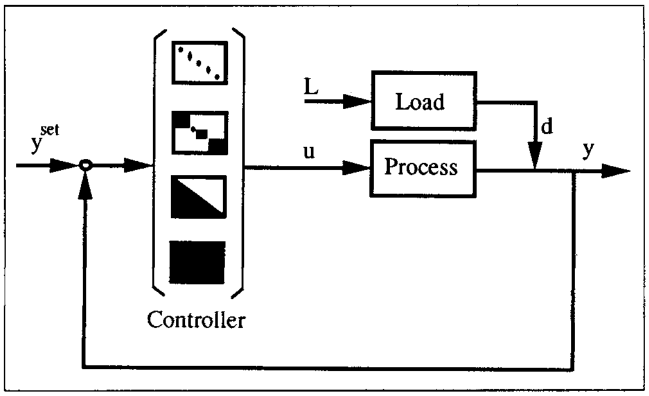

Figure 1. Different controller structures in the conventional feedback system.

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control at steady-state and single-loop control. It can be used to analyze the load responses of interacting design (multiloop SISO controllers) and decoupled design (inverse-based multivariable controller). More importantly, RDG addresses the load rejection problem based on the process and load transfer functions only. Notice that the analyses of Niederlinski (1971) and Stanley et al. (1985) address two controller structure extremes: multiloop SISO controllers and full multivariable controller. As pointed out by Nett and Spang (1987) there is a missing link in the evolution of modern control theories, since all too often only the extreme controller structures, the fully centralized (full multivariable), and the fully decentralized (multiloop SISO) structures, are discussed and analyzed. Many alternative controller structures, for example, the structures between these two extremes, are missing in the analyses. If the controller structure does affect the disturbance rejection capability, then it becomes obvious that any viable tool for this type of analysis should be able to handle all possible controller structures.

The objective of this work is to explore the disturbance rejection capability of all possible controller structures for multivariable systems in a systematic manner. This paper is organized as follows. The controller structures are defined via the internal model control (IMC) (Garcia and Morari, 1982). The generalized relative disturbance gain (GRDG) is defined as the ratio of the net load effect (as the result of interaction) over the original load effect for the controller structure of interest. The physical interpretation becomes obvious using this definition. Similarities and differences between GRDG and RDG are also explored. Note that the controller structure can be of any structure, for example, diagonal, block diagonal (bd), triangular (one-way decoupling) (Arkun et al., 1984) or full controller structure. Next, the relative disturbance gain array (RDGA) is defined. One would be able to evaluate all possible controller structures using this array alone. This greatly simplifies the effort in the synthesis of the best controller structure. GRDG gives some information in the quantitative aspects and its implication becomes obvious after the quantitative analyses. The servo problem can also be treated as a special case of load rejection problem using GRDG. Finally, static GRDG is extended to a frequency-dependent measure and the usefulness of the dynamic GRDG is also explored.

Controller Structures via IMC

For multivariable control system, the controller can have several possible structures, as shown in Figure 1. For example, decouplers can be designed to eliminate interactions among process variables (Luyben, 1973; Arkun et al., 1984). This, generally, results in a full multivariable controller. However, for this noninteracting design with the same controller structure, the decoupler can be an ideal decoupler or a simplified decoupler. Another possibility is one-way decoupling which gives a controller in the triangular structure. We can also design controllers in the block diagonal structure or the diagonal structure. Despite the differences in the design steps and/or structures, one thing is certain that the controller is characterized by the inversion of all or part of the process model. In order to describe the controller structure in a systematic manner, IMC structure (Morari and Zafiriou, 1989) is employed.

Consider the IMC configuration, as shown in Figure 2, where G is the process transfer function matrix, Gc is the IMC controller, and $\tilde{G}$ is transfer function matrix of the process model. Assume G is a square and stable transfer function matrix. The relationship between the output y and the load effect d can be derived as follows:

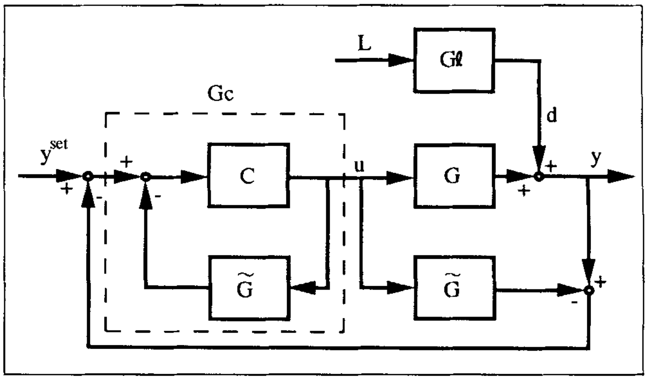


Figure 2. IMC control system.

$$y(s)=G(s)\cdot u(s)+d(s) \tag{1}$$

where u is the manipulated input which can be formulated as:

$$u(s)=[I+Gc(s)(G(s)-\tilde{G}(s))]^{-1}Gc(s)(y^{set}(s)-d(s)) \tag{2}$$

where y^{set} is the set point. When considering the load effect d and substituting Eq. 2 into Eq. 1, we have:

$$\begin{aligned} y(s)&=-G(s)[I+Gc(s)(G(s)-\tilde{G}(s))]^{-1}d(s)+d(s)\\ &=\{I-G(s)\cdot Gc(s)[I+(G(s)-\tilde{G}(s))Gc(s)]^{-1}\}d(s) \end{aligned} \tag{3}$$

Rearranging Eq. 3 gives:

$$y(s)=\{[I-\tilde{G}(s)\cdot Gc(s)][(I+(G(s)-\tilde{G}(s))Gc(s))]^{-1}\}d(s) \tag{4}$$

Express d in terms of a specific load variable, L, and the corresponding load transfer function vector, $G\ell$:

$$y(s)=\{[I-\tilde{G}(s)\cdot Gc(s)][(I+(G(s)-\tilde{G}(s))Gc(s))]^{-1}\}G\ell(s)\cdot L \tag{5}$$

Ideally, one would like to design $Gc(s)$ such that:

$$Gc(s)=\tilde{G}(s)^{-1} \tag{6}$$

This particular choice is not feasible due to the presence of nonminimum phase elements in $\tilde{G}$, for example, right half plane zero or time delays. For implementation, in order to avoid physical unrealizability, $\tilde{G}(s)$ is factorized into the invertible part, $\tilde{G}_-(s)$, and the noninvertible part, $\tilde{G}_+(s)$, with $\tilde{G}_+(0)=I$. That is:

$$\tilde{G}(s)=\tilde{G}_-(s)\cdot\tilde{G}_+(s) \tag{7}$$

Typically, the noninvertible part $\tilde{G}_+(s)$ is limited to the diagonal form. A diagonal filter, $F(s)$, is added to make $Gc(s)$ proper. Therefore, the IMC controller has the form

$$Gc(s) = \tilde{G}_-(s)^{-1} \cdot F(s) \tag{8}$$

Generally, the filter has the form

$$F(s) = \text{diag}\left(\frac{1}{(\epsilon_i s + 1)^{\kappa_i}}\right) \tag{9}$$

where ϵ_i is the tuning constant which should be selected large enough so that robust stability is guaranteed. κ_i is a positive integer.

The relationships between IMC controller, Gc, and conventional feedback controller, C, (Figure 1) can be expressed as:

$$Gc(s) = C(s)(I + \tilde{G}(s)C(s))^{-1} \tag{10}$$

and

$$C(s) = Gc(s)(I - \tilde{G}(s)Gc(s))^{-1} \tag{11}$$

From Eq. 11, it becomes obvious that the structure of the conventional controller $C(s)$ can be obtained from the IMC controller $Gc(s)$. Specially, if $Gc(s)$ is designed according to Eq. 8, we have

$$C(s) = \tilde{G}_-(s)^{-1} F(s)(I - \tilde{G}_+(s) \cdot F(s))^{-1} \tag{12}$$

From Eq. 12, we know the controller structure is determined according to the structure of the inverse of process model, that is, $\tilde{G}_-(s)^{-1}$ since $F(s)$ and $\tilde{G}_+(s)$ are diagonal matrices. Unlike the conventional design philosophy, here, $\tilde{G}(s)$ can be of any structure, for example, $\tilde{G}(s) = G_{diag}(s)$ (a diagonal matrix), $\tilde{G}(s) = G_{bd}(s)$ (a block diagonal matrix), $\tilde{G}(s) = G_{tri}(s)$ (a triangular matrix) or $\tilde{G}(s) = G_{full}(s)$ (a full matrix). The only limitation for Eq. 12 is $\det(\tilde{G}_-(s)) \neq 0$. Note that when $\tilde{G}$ is chosen as diagonal, block diagonal, triangular or full matrix, the controller structure is exactly the same as the process model. The reason is: the structure of $\tilde{G}(s)$ is the same as $\tilde{G}(s)^{-1}$ for these cases. Note that for some other cases, the structures of $\tilde{G}(s)$ and $\tilde{G}(s)^{-1}$ can be different. However, the controller structure can always be obtained by inverting the process model, $\tilde{G}$. The controller structure explored in this paper is with the aforementioned structure unless otherwise mentioned.

In order to characterize the controller structure, a structure selection matrix, Γ, is defined:

$$\Gamma = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \cdots \gamma_{1n} \\ \gamma_{21} & \gamma_{22} & \cdots \gamma_{2n} \\ \vdots & \cdots\cdots & \cdots \\ \gamma_{n1} & \cdots\cdots & \cdots \gamma_{nn} \end{bmatrix} \tag{13}$$

where the ijth entry γ_{ij} is defined as:

$$\gamma_{ij} = \begin{cases} 1, & \text{element picked up (for the controller structure).} \\ 0, & \text{element ignored.} \end{cases}$$

To specify a block diagonal controller structure, some notations are necessary. Let I_k be a subset of the integer set $N = \{1, 2, \cdots, n\}$ with m elements ($1 \leq m \leq n$), that is, $I_k = \{n_1, n_2, \cdots, n_m\}$. For example, for a 4×4 system, we have $N = \{1, 2, 3, 4\}$ and a three-element ($m = 3$) subset can be $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ or $\{2, 3, 4\}$. A block diagonal controller structure with ℓ blocks can be expressed as G_{bd} containing principal block diagonal elements with indices belonging to the subsets I_k ($k = 1, 2, \cdots, \ell$) taken from G and zero elsewhere. As stated earlier, the controller structure can be characterized with process model in the IMC scheme. Mathematically, a specific structure can be expressed as:

$$\tilde{G} = G \otimes \Gamma \tag{14}$$

where $\otimes$ denotes the element-by-element multiplication. Therefore, the controller structure can be characterized by Γ.

It may look strange that deliberate plant/model mismatch ($\tilde{G} \neq G$) is allowed. In the case of full model ($\tilde{G} = G$), Eq. 5 becomes:

$$y = (I - \tilde{G}(s) \cdot Gc(s))G\ell(s) \cdot L \tag{15}$$

Since the term $(I - \tilde{G}(s) \cdot Gc(s))$ can be viewed as the part of integral action $Gc(0) = \tilde{G}(0)^{-1}$ and the specified performance, the noninteracting design ($\tilde{G} = G$) does not provide any change in the load effect $G\ell \cdot L$. This is decoupled design allowing the load effects to come into the multivariable system as it is (as the open-loop load effects). Notice that the noninteracting or decoupled design is referred to responses with respect to set point changes. According to Eq. 5, it is possible to modify the load effect by deliberately introducing plant/model mismatch $\tilde{G} \neq G$. This is, as the result of interaction, the load effect can be modified as:

$$[I + (G(s) - \tilde{G}(s))Gc(s)]^{-1} G\ell(s) \cdot L \tag{16}$$

Here, the difference in G and $\tilde{G}$ can amplify or suppress the load effect. Therefore, the choice of $\tilde{G}$ does affect the disturbance rejection capability. By appropriate selection of $\tilde{G}$, it is possible to achieve better disturbance rejection. Since $\tilde{G}$ characterizes the controller structure (the structure of C), it becomes obvious that the controller structure plays an important role in the disturbance rejection.

Generalized Relative Disturbance Gain

Definition

As mentioned earlier, the effect of load changes can be suppressed or amplified via system interaction. It then becomes clear that a noninteracting control system (an inverse-based multivariable controller) is not necessary for the best controller structure as far as the disturbance rejection is concerned. That is, on some occasions, interaction is necessary for load rejection. Therefore, a measure for the evaluation of the controller structure vs. disturbance rejection capability is desirable. Eq. 5 provides the basis for analyzing the load effects under different controller structures. The second factor (in the bracket) in the RHS (right-hand-side) of Eq. 5 shows the net load effect as the result of possible system interaction (as the consequence of plant/model mismatch, that is, $\tilde{G} \neq G$). Since controller parameters in Gc are involved in Eq. 5, a logical approach is assuming $Gc(s) = \tilde{G}(s)^{-1}$ (a perfect controller). Therefore, the net load effect $G\ell^*$ can be expressed as:

$$G\ell^*(s) = [I + (G(s) - \tilde{G}(s))\tilde{G}(s)^{-1}]^{-1} G\ell(s)$$
$$= \tilde{G}(s) G(s)^{-1} G\ell(s) \quad (17)$$

Eq. 17 shows how the original load effect can be modified with the structure characterized by $\tilde{G}$ and the net effect is simply $\tilde{G}\cdot G^{-1}\cdot G\ell$. If the noninteracting design, $\tilde{G}=G$, is preferred, then the net load effect becomes:

$$G\ell^*(s) = G(s)G(s)^{-1}G\ell(s) = G\ell(s) \quad (18)$$

That is, for a decoupled control system, the net load effect remains the same as open-loop one. In this section, only steady-state aspects are considered. The symbol (s) is dropped for clarity.

An interaction measure GRDG is defined to evaluate the load effect under a specific controller structure (closed-loop load effect) over the open-loop load effect. Conceptually, it is:

$$\text{GRDG} = \frac{\text{closed-loop load effect}}{\text{open-loop load effect}} \quad (19)$$

GRDG is a vector which is a function of G, $\tilde{G}$ and $G\ell$. Mathematically, it becomes:

$$\text{GRDG}\{G, \tilde{G}, G\ell\} = G\ell^* \oslash G\ell = (\tilde{G}\cdot G^{-1}\cdot G\ell)\oslash G\ell \quad (20)$$

where $\oslash$ denotes element-by-element division. GRDG is a vector with element $\delta_i = g\ell_i^*/g\ell_i$ where $g\ell_i^*$ is the ith element of $G\ell^*$ and $g\ell_i$ is the ith element of $G\ell$. Physically, GRDG measures the net load effect of a *specific* controller structure over the open-loop load effect. Actually, the open-loop load effect is the same as the load effect when the full model is employed in $\tilde{G}$ (a full multivariable controller), as shown in Eq. 18. This definition is very similar to the relative gain array (RGA) except that the major concern here is the load disturbance. Furthermore, GRDG is defined according to a specified controller structure. The controller structure can take any form of interest except for the ratio schemes. This goes beyond the structure limitation imposed on RGA, for example, BRG (Block Relative Gain) (Manousiouthakis et al., 1986) is defined for block diagonal structure.

The entry of GRDG (for example, $\text{GRDG} = [\delta_1, \delta_2, \cdots, \delta_n]^T$) lies between $-\infty$ to ∞. For example, when $g\ell_i = 0$ and $g\ell_i^* > 0$, then δ_i goes to infinity. Similar to many popular measures it is also a scaling independent measure as shown in the next property.

Property 1. The GRDG is invariant under both input and output scaling. By input scaling we mean post-multiplication of G by a nonsingular diagonal matrix S_I and $G\ell$ by a nonzero scalar s_i. By output scaling we mean premultiplication of G and $G\ell$ by a nonsingular diagonal matrix S_0.

Proof. Let G and $\tilde{G}$ be premultiplied and postmultiplied by diagonal matrices S_0 and S_I respectively and $G\ell$ be premultiplied and postmultiplied by S_0 and s_i, respectively.

$$G' = S_0\cdot G\cdot S_I$$
$$\tilde{G}' = S_0\cdot \tilde{G}\cdot S_I$$
$$G\ell' = S_0\cdot G\ell\cdot s_i$$

Therefore, the scaled GRDG can be written as

$$\begin{aligned}\text{GRDG}' &= (\tilde{G}'\cdot(G')^{-1}\cdot G\ell')\oslash G\ell' \\ &= [S_0\cdot\tilde{G}\cdot S_I\cdot(S_0\cdot G\cdot S_I)^{-1}\cdot(S_0\cdot G\ell\cdot s_i)]\oslash[S_0\cdot G\ell\cdot s_i] \\ &= (S_0\cdot\tilde{G}\cdot G^{-1}\cdot G\ell)\oslash(S_0\cdot G\ell) \\ &= (\tilde{G}\cdot G^{-1}\cdot G\ell)\oslash G\ell \\ &= \text{GRDG}\end{aligned}$$

Relationship between GRDG and RDG

In the literature, only two interaction measures (Stanley et al., 1985; Skogestad and Morari, 1987), based on the process and load transfer functions, are proposed in dealing with load disturbances. The disturbance condition number of Skogestad and Morari (1987) is capable of discriminating alternative control structure (selection of manipulated and controlled variables) from the viewpoint of disturbance direction. The RDG of Stanley et al. (1985) can be used to analyze disturbance rejection capability of different controller structures. Similarities and differences between RDG and GRDG are analyzed.

Consider the following multivariable process:

$$y = G\cdot u + G\ell\cdot L \quad (21)$$

where y is the output, u is the manipulated input and L is the load variable. The ith element of RDG is defined as (Stanley et al., 1985):

$$\beta_i = \frac{\left[\dfrac{\partial u_i}{\partial L}\right]_{y_j}}{\left[\dfrac{\partial u_i}{\partial L}\right]_{y_i, u_{j,j\neq i}}} \quad (22)$$

The term in the numerator denotes the change in the manipulated variable u_i needed for perfect disturbance rejection. The term in the denominator represents the change in manipulated variable u_i when one of the output y_i is kept perfect. Equation 22 can be rearranged and the vector of RDG can be expressed as:

$$\text{RDG}\{G, G_{\text{diag}}, G\ell\} = (G^{-1}\cdot G\ell)\oslash((G_{\text{diag}})^{-1}\cdot G\ell) \quad (23)$$

where $G_{\text{diag}} = \text{diag}(g_{ii})$. Since G_{diag} is a diagonal matrix, Eq. 23 can be rewritten as

$$\text{RDG} = (G_{\text{diag}}\cdot G^{-1}\cdot G\ell)\oslash G\ell \quad (24)$$

Comparing Eq. 24 with Eq. 20, one finds that RDG is exactly the GRDG when a fully decentralized (diagonal) controller ($\tilde{G} = G_{\text{diag}}$) is employed. This finding is consistent with the argument of Stanley et al. (1985): RDG measures the disturbance rejection capability of interacting design (multiloop SISO controllers). Despite the difference in the forms of expressions between RDG and GRDG (Eq. 24 and Eq. 20), RDG measures the net load effects of two specific controller structures: multiloop SISO controllers and full multivariable controller. Equation 19 provides an alternative interpretation for RDG.

It becomes obvious that RDG is extended to handle different controller structures via the introduction of GRDG. In addition to the diagonal controller, controllers of different form can also be analyzed using GRDG. Furthermore, it possesses the good properties of RDG such as a dimensionless and scaling independent measure.

RDGA and Controller Structures

The GRDG defined in Eq. 20 is capable of analyzing the disturbance rejection capability of a specific controller structure. Unfortunately, it is pairing as well as structure dependent. This is the value of δ_i which depends on the controller structure. Moreover, for the same structure, δ_i changes with the input-output pairing. Therefore, in the synthesis of controller structure, one has to go over all possible controller structures under all possible pairings to compute δ_i. This practice can be simplified greatly with the introduction of RDGA.

RDGA

The concept of RDGA is, again, very similar to RGA (Bristol, 1966) except that the phenomena of interest is load disturbance. Since RDG is pairing dependent (β_i depends on the input-output pairing) (Stanley et al., 1985; Skogestad and Morari, 1987), a $n\times n$ array can be constructed after going through n possible pairings (forming n vectors). Therefore, an augmented version of relative disturbance gain β_{ij} can be defined and a matrix can be formed. The matrix RDGA ($\mathcal{B}$) is defined as:

$$\mathcal{B} = \begin{array}{c} \begin{array}{cccc} u_1 & u_2 & \cdots & u_n \end{array} \\ \begin{bmatrix} \beta_{11} & \beta_{12} & \cdots & \beta_{1n} \\ \beta_{21} & \beta_{22} & & \beta_{2n} \\ \cdot & \cdots & & \cdot \\ \beta_{n1} & \cdots & \cdots & \beta_{nn} \end{bmatrix} \end{array} \begin{array}{c} \\ y_1 \\ y_2 \\ \cdot \\ y_n \end{array} \tag{25}$$

The ijth entry β_{ij} corresponds to the RDG when the ith ouput is paired with the jth input. Notice that, since the vector $G\ell$ is involved in the computation of β_{ij}, corresponding changes in $G\ell$ have to be made for any permutation in the outputs in the matrix G. As pointed out by Stanley et al. (1985), RDG is related to the diagonal elements of RGA. Here, relationship between RDGA and RGA can be derived from Eq. 20 with $\tilde{G} = G_{\text{diag}}$.

$$\beta_{ij} = \lambda_{ij} + \sum_{\substack{k=1 \\ k\neq i}}^{n} \frac{g_{ij}\hat{g}_{jk}g\ell_k}{g\ell_i} \tag{26}$$

where λ_{ij} is the ijth element of RGA of G and $\hat{g}_{ik}$ is the ikth element of G^{-1}. The RGA of G is defined as (Bristol, 1966)

$$\text{RGA}\{G\} = G \otimes (G^{-1})^T \tag{27}$$

In a matrix notation, the $\mathcal{B}$ matrix can be promptly calculated from

$$\mathcal{B} = [G^{-1}\cdot \text{vdm}(G\ell)]^{-1}\cdot[\text{vdm}(G^{-1}G\ell)] \tag{28}$$

Note that the operator vdm($\cdot$) transforms a vector ($\cdot$) into a matrix with element put on the corresponding diagonal position, that is, the ith element of a vector ($\cdot$) is put in the iith entry of a matrix. Equation 28 simplifies the computation of RDGA. The $\mathcal{B}$ matrix has several important properties which are useful for the selection of controller structure. The first property is that the $\mathcal{B}$ matrix is invariant under both input and output scaling. This result is obvious since RDG is scaling independent (Stanley et al., 1985).

Property 2. Any permutation of columns and/or rows in G results in the same permutation in the $\mathcal{B}$. Notice that any permutation of rows in G leads to corresponding permutation of $G\ell$, but this is not true for the permutation of columns in G. Mathematically, if $G_P = P_0 G P_I$ and $G\ell_P = P_0 G\ell$ with P_I and P_0 being two square permutation matrices, then $\mathcal{B}_P = P_0\,\mathcal{B} P_I$.

Proof: P_0 and P_I are $n\times n$ permutation matrices. Since P_0 and P_I are orthogonal matrices, then $P_0^{-1} = (P_0)^T$ and $P_I^{-1} = (P_I)^T$. From Equation 28 and Appendix A, the permuted $\mathcal{B}$ becomes

$$\begin{aligned} \mathcal{B}_P &= [(P_0GP_I)^{-1}\cdot \text{vdm}(P_0G\ell)]^{-1}\cdot[\text{vdm}((P_0GP_I)^{-1}(P_0G\ell))] \\ &= [P_IG^{-1}\text{vdm}(G\ell)(P_0)^T]^{-1}\cdot[P_I^{-1}\cdot\text{vdm}(G^{-1}G\ell)(P_I^{-1})^T] \\ &= P_0[G^{-1}\cdot\text{vdm}(G\ell)]^{-1}\cdot[\text{vdm}(G^{-1}G\ell)]P_I \\ &= P_0\mathcal{B}P_I \end{aligned}$$

Therefore, similar to RGA, the permutations in the inputs and outputs correspond to the same permutations in RDGA. This implies that we can evaluate the disturbance rejection capabilities of all possible pairings under multiloop SISO controllers by rearranging $\mathcal{B}$. Since several other issues such as stability and failure tolerance (Yu and Fan, 1990; Chang and Yu, 1991) are related to the selection of an appropriate pairing, the relationship between pairing and disturbance rejection will not be discussed in this paper. The controller structure selection under a specific pairing for better disturbance rejection is of interest.

The next property of RDGA is, again, very similar to that of RGA.

Property 3. The row sum of RDGA is equal to 1.

Proof: From Eq. 26,

$$\begin{aligned} \sum_{j=1}^{n}\beta_{ij} &= \sum_{j=1}^{n}\left(\lambda_{ij} + \sum_{\substack{k=1\\k\neq i}}^{n}\frac{g_{ij}\hat{g}_{jk}g\ell_k}{g\ell_i}\right) \\ &= 1 + \sum_{j=1}^{n}\sum_{\substack{k=1\\k\neq i}}^{n}\frac{g_{ij}\hat{g}_{jk}g\ell_k}{g\ell_i} \\ &= 1 + \sum_{\substack{k=1\\k\neq i}}^{n}\frac{0\cdot g\ell_k}{g\ell_i} \\ &= 1 \end{aligned}$$

The implications of this property will be discussed later. It is clear that the introduction of RDGA ($\mathcal{B}$) solves at least part of the computational problem: different pairings can be evaluated by rearranging $\mathcal{B}$.

Controller structure by RDGA

Again, similar to the relationship between RGA and BRG (Manousiouthakis et al., 1986), for example, the diagonal element of BRG is the summation of relative gain in the corresponding block, RDGA possesses similar property and beyond. If the structure of $\tilde{G}$ is specified, as shown in Figure 1, the GRDG for the ith output δ_i is simply the rowwise summation of RDGA with corresponding structure. The structure is not limited to the block diagonal structure (as that between BRG and RGA). It can be of *any* possible structure. Mathematically, this is (see Appendix B for the proof):

$$\delta_i = \sum_{j=1}^{n} \beta_{ij}\gamma_{ij} \tag{29}$$

This is a very important result, since we are able to evaluate all structures with all pairings based on RDGA alone. For example, the elements of GRDG for a diagonal controller ($\tilde{G} = G_{\text{diag}}$) are simply the diagonal elements of RDGA, that is, $\delta_i = \beta_{ii}$ for all i's. The GRDG for block diagonal controller ($\tilde{G} = G_{bd}$) is equal to the rowwise summation of RDGA in the corresponding kth block, that is,

$$\delta_i = \sum_{j=1}^{m} \beta_{i,n_j},\ n_j \epsilon I_k.$$

For a triangular controller structure, $\tilde{G} = G_{tri}$, the δ_i is simply the row sum of RDGA for that particular triangle. For the full controller structure, the δ_i's are equal to one for all i's (Property 3). This corresponds to the decoupled design (from the definition of GRDG). Certainly, some of the outputs can be decoupled by choosing full elements in the corresponding rows. This implies the system can be decoupled on the individual output basis. More importantly, all these structures can be evaluated from $\mathcal{B}$ alone.

Example 1. Assume

$$G = \begin{bmatrix} 1.0 & 0.1 & 0.3 \\ 0.2 & 1.0 & -0.5 \\ 0.5 & 0.3 & 1.0 \end{bmatrix} \text{ and } G\ell = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.2 \end{bmatrix}$$

From Eq. 28, $\mathcal{B}$ becomes:

$$\mathcal{B} = \begin{bmatrix} 0.44 & 0.24 & 0.32 \\ 0.04 & 1.22 & -0.26 \\ 0.11 & 0.37 & 0.52 \end{bmatrix}$$

If the structure selection matrix Γ is specified as:

$$\Gamma = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

we can compute GRDG in the following ways:

(i) From Eq. 20

$$\text{GRDG} = [0.44 \quad 0.96 \quad 1.00]^T$$

(ii) From Eq. 29

$\delta_1 = 0.44$, $\delta_2 = 1.22 + (-0.26) = 0.96$ and

$$\delta_3 = 0.11 + 0.37 + 0.52 = 1.0$$

GRDG of any structures can be calculated similarly. The properties of RDGA presented here simplify the effort in the synthesis of the controller structure. The disturbance rejection capabilities of all structures can be checked by rearranging $\mathcal{B}$ or by adding up the elements in $\mathcal{B}$.

Analysis and Synthesis

Quantitatively the controller structure gives a small value of δ_i and is preferable, since δ_i is the ratio of the net load effect ($g\ell_i^*$; as the result of system interaction) over the open-loop load effect ($g\ell_i$; or the load effect for full multivariable controller). Moreover, GRDG (δ_i's) relates *quantitatively* to the margin of improvement which can be achieved via controller structure selection. Stanley et al. (1985) gives quantitative analysis of RDG based on integrated error (IE).

Analysis

The GRDG is related quantitatively to the integrated error, area under response curve. The IE (a vector) of the output, y, under a specific controller structure, is defined as

$$\text{IE}\{\tilde{G}\} = \int_0^\infty (y^{\text{set}} - y(t))dt = \int_0^\infty e(t)dt \tag{30}$$

In terms of Laplace transformed variable, IE can be expressed as

$$\text{IE}\{\tilde{G}\} = \lim_{s\to 0} E(s) \tag{31}$$

where $E(s)$ is the Laplace transformation of $e(t)$. From Eqs. 5, 9, and 20, ratios of IE's ($\text{IE}\{\tilde{G}\} \oslash \text{IE}\{G\}$) can be expressed as (see Appendix C for the derivation):

$$\text{IE}\{\tilde{G}\} \oslash \text{IE}\{G\} = (\tilde{\mathcal{E}} \oslash \mathcal{E}) \cdot \text{GRDG} \tag{32}$$

where $\tilde{\mathcal{E}}$ and $\mathcal{E}$ can be considered as diagonal tuning matrices, for example, $\mathcal{E} = \text{diag}(\kappa_i \epsilon_i)$. Similar results were derived by Stanley et al. (1985) for multiloop SISO controllers. If the closed-loop time constants and the orders of the filters are assumed to be the same, for example, $\tilde{\epsilon}_i = \epsilon_i$ and $\tilde{\kappa}_i = \kappa_i$, then we have:

$$\text{IE}\{\tilde{G}\} \quad \text{IE}\{G\} = \text{GRDG} \tag{33}$$

Under these assumptions, GRDG is exactly the ratio of IE's for the structured and full multivariable controllers. The value of δ_i does compare the load rejection capabilities for systems with different controller structures. For a more general case (less restrictive conditions on Gc), δ_i can be viewed, at least in an engineering sense, as an approximation of margin of improvement. Therefore, a structure giving small value in δ_i is preferred (at least from the IE standpoint).

Synthesis

The RDGA provides a simple way for the synthesis of con-

Table 1. Process Transfer Function Matrices and Load Transfer Function Vectors for *DL* Column and *CL* Column

$G(s)$				$G\ell(s)$
		(*A*) *DL Column*		
$\frac{-11.5e^{-s}}{(23s+1)(5s+1)}$	0.0	0.0		$\frac{-1.95e^{-5s}}{(12s+1)^2}$
$\frac{3.75e^{-2s}}{(14s+1)(3s+1)^2}$	$\frac{1.6e^{-1.3s}}{(13s+1)(3s+1)}$	$\frac{-1.2e^{-1.05s}}{(15.5s+1)(3s+1)}$		$\frac{1.52e^{-6s}}{(12s+1)^2(5s+1)}$
$\frac{20.6e^{-1.9s}}{(23s+1)(18s+1)}$	$\frac{-7.5e^{-2.3s}}{(37.3s+1)(2s+1)}$	$\frac{23.1e^{-s}}{(42s+1)(2s+1)}$		$\frac{-4.45e^{-7s}}{(40s+1)(10s+1)^2}$
		(*B*) *CL Column*		
$\frac{4.45}{(14s+1)(4s+1)}$	$\frac{-7.4}{(16s+1)(4s+1)}$	0.0	$\frac{0.35}{(25.7s+1)(2s+1)}$	$\frac{1.02e^{-4.5s}}{(25s+1)(2s+1)^2}$
$\frac{17.3e^{-0.9s}}{(17s+1)(0.5s+1)}$	$\frac{-41}{(21s+1)(s+1)}$	0.0	$\frac{9.2e^{-0.3s}}{20s+1}$	$\frac{19.7e^{-0.3s}}{(25s+1)(s+1)}$
$\frac{0.22e^{-1.2s}}{(17.5s+1)(4s+1)}$	$\frac{-4.66}{(13s+1)(4s+1)}$	$\frac{3.6}{(13s+1)(4s+1)}$	$\frac{0.042(78.7s+1)}{(21s+1)(11.6s+1)(3s+1)}$	$\frac{0.75e^{-5s}}{(15.6s+1)(2s+1)}$
$\frac{1.82e^{-s}}{(21s+1)(s+1)}$	$\frac{-34.5}{(20s+1)(s+1)}$	$\frac{12.2e^{-0.9s}}{(18.5s+1)(s+1)}$	$\frac{-6.92e^{-0.6s}}{20s+1}$	$\frac{16.61e^{-0.6s}}{(25s+1)(2s+1)}$

Table 2. GRDG for *DL* Column

Controller Structure	GRDG
Diagonal	$(1.00\ \ 0.41\ \ 1.13)^T$
bd[(1,2),3]	$(1.00\ \ 0.83\ \ 1.13)^T$
bd[(1,3),2]	$(1.00\ \ 0.41\ \ 0.34)^T$
bd[(2,3),1]	$(1.00\ \ 0.58\ \ 1.79)^T$
Full	$(1.00\ \ 1.00\ \ 1.00)^T$

troller structure. Since δ_i corresponds to the IE of a structured controller, we are able to synthesize the controller structure according to δ_i. Let us take an example studied by Ding and Luyben (1990) to illustrate this. Table 1A gives the process transfer function matrix ($G(s)$) and the load transfer function vector ($G\ell(s)$) of *DL* column. For a fixed pairing, the RDGA can be calculated from the steady-state gains in Table 1A.

$$\mathcal{B} = \begin{array}{c} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} \\ \begin{bmatrix} 1.00 & 0. & 0. \\ 0.42 & 0.41 & 0.17 \\ -0.79 & 0.66 & 1.13 \end{bmatrix} \end{array} \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array}$$

For the purpose of illustration, let us limit the controller structures to be of diagonal, block diagonal (bd), and full multivariable structures. The synthesis of the controller structure of a better disturbance rejection can be done in a straightforward manner. For the diagonal controller ($\tilde{G}=G_{\text{diag}}$), the GRDG is simply the diagonal element of $\mathcal{B}$ (Table 2). For the full multivariable controller, δ_i is 1 for all i's (Table 2 or Property 3). For the block diagonal structure, let bd[(1,2),3], bd[(1,3),2], and bd[1,(2,3)] denote the block diagonal controller with [(1,2),3], [(1,3),2], and [1,(2,3)] structure respectively. The GRDG for the block diagonal controller can be calculated from $\mathcal{B}$ by adding up the elements in the corresponding block. For example, the GRDG for the bd[(1,2),3] structure is: $\delta_1=1.00+0.$, $\delta_2=0.42+0.41$ and $\delta_3=1.13$. The GRDG's for these block diagonal structures are given in Table 2. From Table 2, we can evaluate the disturbance rejection capabilities of these controller structures based on the IE (Eq. 29) analysis. Table 2 shows that the block diagonal controller bd[(1,3),2] offers the best disturbance rejection capability, for example, GRDG = $[1.00, 0.41, 0.34]^T$. Roughly, 60% improvement in IE's over the full multivariable controller can be made for the outputs y_2 and y_3. Simulation results, as shown in Figure 3, also show that for a step load change the block diagonal controller bd[(1,3),2] performs better than the full multivariable controller. In the simulation, for simplicity, the IMC controller with $Gc=\tilde{G}(0)^{-1}$ is used. Note that the controller structure in the conventional feedback structure, as shown in Figure 1, is the same as the structure of $\tilde{G}$.

Let us take a 4×4 system as another example. This is a heat-integrated distillation column (*CL* column) studied by Chiang and Luyben (1988). The process and load transfer functions are given in Table 1B. The RDGA is:

$$\mathcal{B} = \begin{bmatrix} -6.43 & 7.28 & 0. & 0.15 \\ -1.29 & 2.09 & 0. & 0.20 \\ -0.43 & 6.23 & -4.83 & 0.03 \\ -0.16 & 2.08 & -0.74 & -0.18 \end{bmatrix}.$$

For the diagonal controller, the GRDG is: $\delta_1=-6.43$, $\delta_2=2.09$, $\delta_3=-4.83$ and $\delta_4=-0.18$. Note that there are several negative elements in δ_i's. For example for the output y_4, the δ_i is -0.18 which is a rather small number (far less than 1). This implies that the IE for the diagonal controller is roughly 20% of that for the full multivariable controller. However, the theorem in Appendix D reveals that, for a system with a negative δ_i, the IE is not an adequate measure of performance. The reason is when $\delta_i<0$, cancellation occurs in calculating the area under response curve (y_i crossing the set point and IE is the sum of the positive and negative areas). Therefore, if no further information is available, for example, dynamic information, a structure with a negative δ_i (even a small number) is generally not preferred. Furthermore, it should be noticed that the can-

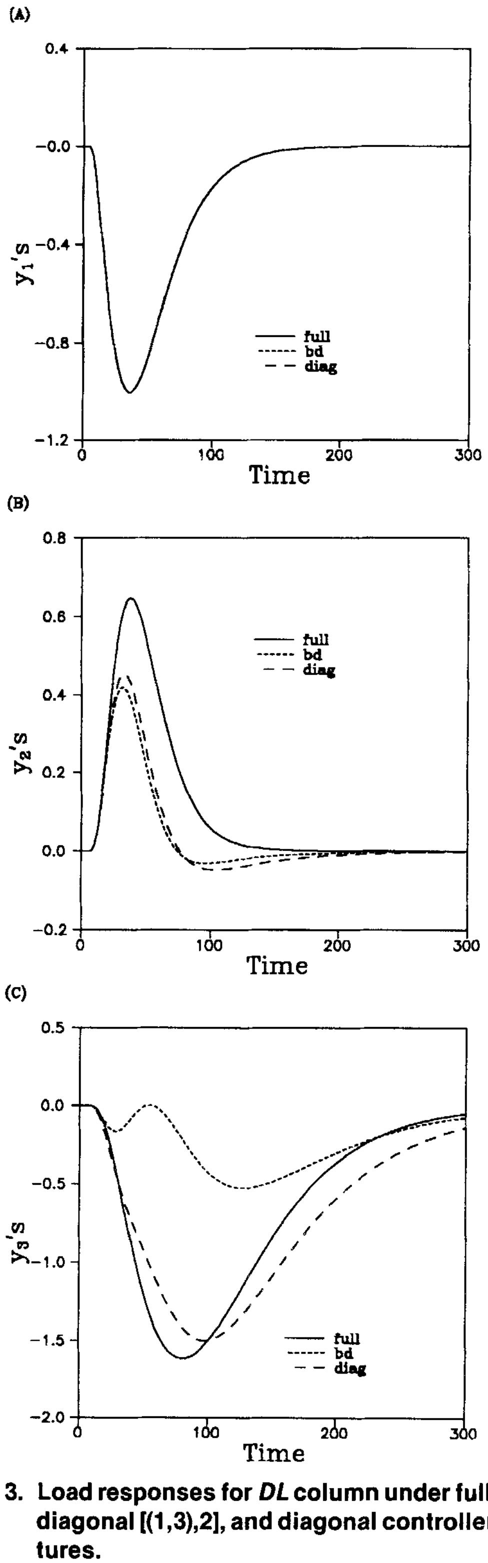


Figure 3. Load responses for *DL* column under full, block diagonal [(1,3),2], and diagonal controller structures.

cellation may also occur even when $\delta_i > 0$. This will be shown later in Example 3. It should be emphasized that, up to this point, only steady-state gains are needed for the analysis. For the *CL* column, δ_4 is -0.18 which is a perfect choice for disturbance rejection based on IE (at least from the y_4 point of view). The IE's calculated from simulations, as shown in Figure 4, for the diagonal and full multivariable controllers also confirm this. For y_4 the ratio of IEs is:

$$\frac{\mathrm{IE}\{G_{\mathrm{diag}}\}}{\mathrm{IE}\{G_{\mathrm{full}}\}} = -0.17 \cong \delta_4$$

However, from the load responses, shown in Figure 4, one can see that the margin of improvement for diagonal controller in y_4 is not quite as large as the one promised from the IE analysis. If integrated absolute error (IAE) is used as the performance measure, the result for y_4 becomes:

$$\frac{\mathrm{IAE}\{G_{\mathrm{diag}}\}}{\mathrm{IAE}\{G_{\mathrm{full}}\}} = 0.58$$

Despite the fact that the improvement in the IAE can still be made with the diagonal controller in this example. Without further dynamic information, the structure with a negative δ_i should be avoided. The reason is the response curves (for example, y_4 in Figure 4) can take many possible forms (for example, the cancelled area can be arbitrarily large). Therefore, another principle in the synthesis of the controller structure (from RDGA) is: avoid the controller structure with a negative δ_i. It should be emphasized that this rule can be applied when no further dynamic information is available. If the frequency responses of $|\delta_i(i\omega)|$ is favorable (as will be discussed subsequently), then a structure with $\delta_i(0) < 0$ can also be considered. For *CL* column, the optimal controller structure for disturbance rejection can be synthesized based on these principles in a straightforward manner. In terms of Γ, the optimal controller structure is:

$$\Gamma = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

The corresponding GRDG is:

$$\mathrm{GRDG} = [0.85 \quad 0.80 \quad 0.97 \quad 1.00]^T$$

Note that the controller structure is neither in the block diagonal form nor in the form of full matrix. This means the best controller structure for disturbance rejection is not limited to the familiar form and GRDG is capable of analyzing all possible controller structures. Figure 4 shows the load responses for the controller with the optimal structure as well as the diagonal and full multivariable controllers. Results show that improvement can be made over the other two structures, as shown in Figure 4.

Trade-off between load and set point responses

Up to this point, only the load responses are considered based on GRDG. When the set point responses are considered, what is the best controller structure for the set point responses? When both factors are of importance, what is the trade-off between these two responses? Since the set point change can be considered as a special kind of load disturbance, GRDG can also be used to evaluate the set point responses. Actually, the ith unit step set point change can be treated as a load

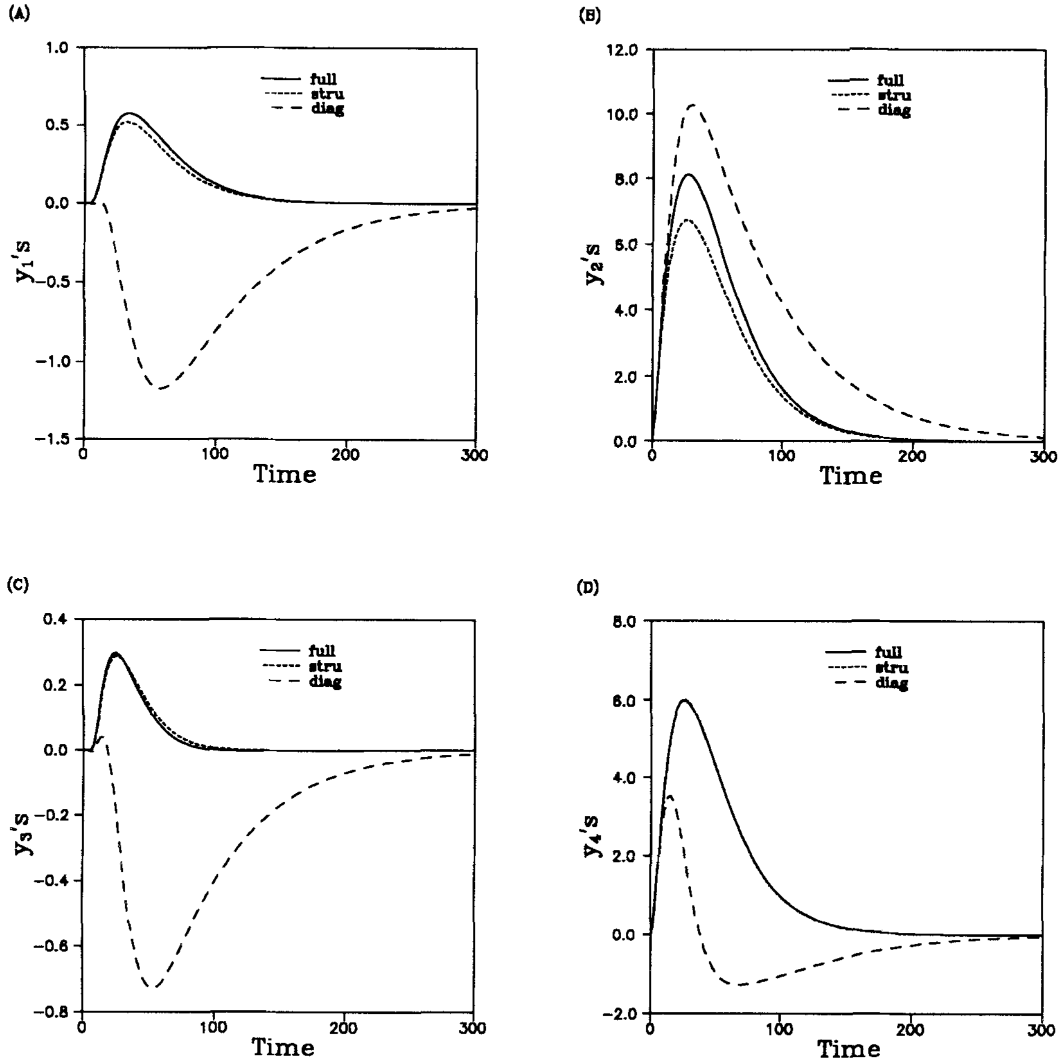


Figure 4. Load responses for *CL* column under full, structured and diagonal controller structures.

disturbance (Skogestad and Morari, 1987). Let z_i be the ith column vector of the identity matrix.

$$z_i = [0 \cdots 0\ 1\ 0 \cdots 0]^T \tag{34}$$

Physically, z_i can be considered as a unit step change in the ith set point. Therefore, the servo problem can be treated directly by replacing $\boldsymbol{G\ell}$ with z_i.

For the diagonal controller, that is, $\tilde{\boldsymbol{G}} = \boldsymbol{G}_{\text{diag}}$, the diagonal elements of the matrix $\tilde{\boldsymbol{G}} \cdot \boldsymbol{G}^{-1}$ are just the diagonal elements of RGA. Therefore, the ith element of GRDG becomes λ_{ii}, that is, $\delta_i = \lambda_{ii}$ for $\boldsymbol{G\ell} = z_i$. This can be seen from Eq. 26. Considering δ_i, we have

$$\delta_i = \lambda_{ii} + \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{g_{ii} \hat{g}_{ik} g\ell_k}{g\ell_i} \tag{35}$$

For ith set point change, $g\ell_i = 1$ and $g\ell_{k,k\neq i} = 0$. From Eq. 35, it becomes obvious that the ith element of GRDG is equal to the iith element of RGA. However, the other elements in GRDG become infinity (∞), since $g_{\ell k} = 0$ for $k \neq i$ (Table 3). The reason is quite clear since the other elements (other than the ith element) in z_i are zero and the division in the definition of GRDG leads to infinity. This does not mean the IE's for the diagonal controller is large. It is because the IE in the denominator (for the full multivariable controller) is zero. Similar results can be derived for the block diagonal controller, $\tilde{\boldsymbol{G}} = \boldsymbol{G}_{bd}$. Here, the ith element of GRDG is related to the block relative gain (BRG) (Manousiouthakis et al., 1986). Let $\tilde{\boldsymbol{G}} = \boldsymbol{G}_{bd}$. The BRG is defined as

$$\text{BRG}\{\boldsymbol{G}_{bd}\} = \boldsymbol{G}_{bd} \cdot (\boldsymbol{G}^{-1})_{bd} \tag{36}$$

Therefore, for the ith set point change, the ith element of GRDG is equal to the iith element of BRG, that is,

$$\delta_i = \text{BRG}_{ii} \tag{37}$$

This also can be calculated from rowwise summation of RGA within the block (Manousiouthakis et al., 1986). All other elements of GRDG, except for δ_i, have the value of infinity.

Table 3. GRDG for a Set Point Change for Different Controller Structures

Set Point Change	GRDG Full Controller	GRDG Block Diagonal Controller	GRDG Diagonal Controller
$\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$	$\begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$	$\begin{bmatrix} \infty \\ \vdots \\ \infty \\ \mathrm{BRG}_{ii} \\ \infty \\ \vdots \\ \infty \end{bmatrix}$	$\begin{bmatrix} \infty \\ \vdots \\ \infty \\ \lambda_{ii} \\ \infty \\ \vdots \\ \infty \end{bmatrix}$

Again, this does not imply the IEs for the block diagonal controller are necessarily large. The reason is simply the IEs are zero for the full multivariable controller. For the full multivariable controller, the GRDG is unity for all δ_i's, as shown in Table 3.

Therefore, for the servo problem, the full multivariable controller (decoupled design) is the best choice when one is evaluating the IEs for *all* variables. Trade-off in the selection of controller structure becomes obvious. One has to evaluate types of disturbances encountered in particular processes of interest. However, more importantly, the servo problem can also be analyzed using GRDG. This servo problem can be considered as another load disturbance.

Extension to Dynamic GRDG

Up to this point, only the steady-state aspects of the GRDG are discussed. On many occasions, dynamic information, for example, transfer functions, is available. We can utilize such information to provide further analyses. Steady-state GRDG can be extended to frequency-dependent GRDG in a straightforward way. The dynamic GRDG is defined as:

$$d-\mathrm{GRDG}(s) = G\ell^*(s) \oslash G\ell(s) \tag{38}$$

The definition here is exactly the same as the one given in Eq. 20 except that $d-\mathrm{GRDG}(s)$ is a function of frequency (by setting $s = i\omega$). However, similar to the situation the dynamic RGA faces (Hovd and Skogestad, 1990), perfect control is assumed in the definition of $G\ell^*(s)$ (Eq. 17). Then, how the measure under the "perfect control" situation can be applied or interpreted in practical applications (under not so "perfect" control).

A starting point for the interpretation of dynamic GRDG is the closed-loop error, since $e_{\mathrm{CL}}(s)$ is the major concern in practical applications. From the IMC structure, $e_{\mathrm{CL}}(s)$ is:

$$\begin{aligned} e_{\mathrm{CL}}(s) &= -G\ell_{\mathrm{CL}}(s) \cdot L \\ &= -(I - G(s)Gc(s))[I + (G(s) - \tilde{G}(s))Gc(s)]^{-1} \\ &\quad \cdot G\ell(s) \cdot L \end{aligned} \tag{39}$$

The open-loop error, on the other hand, is:

$$e_{\mathrm{OL}}(s) = -G\ell(s) \cdot L \tag{40}$$

Therefore, a performance measure S^* can be defined.

$$\begin{aligned} S^*(s) &= e_{\mathrm{CL}}(s) \oslash e_{\mathrm{OL}}(s) \\ &= G\ell_{\mathrm{CL}}(s) \oslash G\ell(s) \end{aligned} \tag{41}$$

$S^*(s)$ is very similar to the sensitivity function (Morari and Zafiriou, 1989) except that the load effect is taken into account. In effect, $S^*(s)$ is the normalized version of TLC (Yu and Luyben, 1986) ($\mathrm{TLC}(s) = G\ell_{\mathrm{CL}}(s)$). As pointed out by Yu and Luyben, TLC is a dynamic measure of load performance. For a specific IMC design (Eq. 8), $S^*(s)$ becomes:

$$\begin{aligned} S^*(s) = |\{ (I - \tilde{G}_+(s)F(s)) \cdot [\tilde{G}(s)G(s)^{-1}(I - \tilde{G}_+(s)F(s)) \\ + \tilde{G}_+(s)F(s)]^{-1} G\ell^*(s) \} \oslash G\ell(s)| \end{aligned} \tag{42}$$

Let us take a 2×2 system to illustrate the implication of dynamic GRDG.

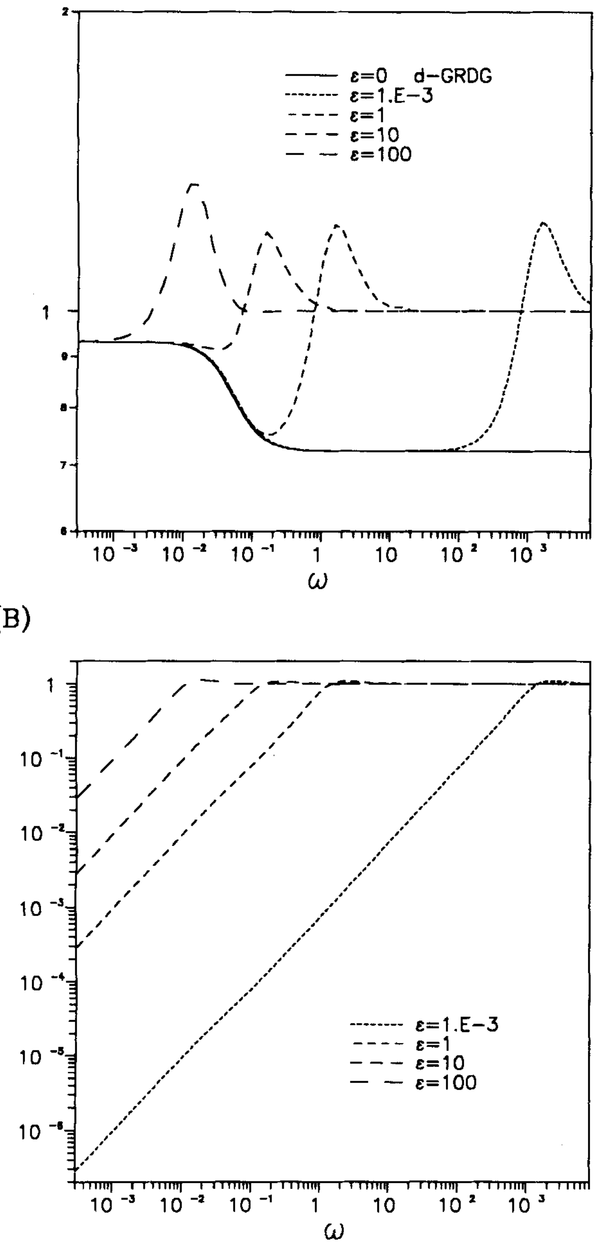
Example 2. Assume

$$G(s) = \begin{bmatrix} \dfrac{1.5}{20s+1} & \dfrac{-0.2}{20s+1} \\ \dfrac{0.7}{10s+1} & \dfrac{0.1}{10s+1} \end{bmatrix} \text{ and } G\ell(s) = \frac{1}{10s+1}\begin{bmatrix} 0.5 \\ 0.2 \end{bmatrix}$$

with the diagonal controller structure $\tilde{G}(s) = G_{\mathrm{diag}}(s)$ and a filter $F = \mathrm{diag}[1/(\epsilon s + 1)]$. Let us ignore the integrating part, that is, considering the term $|\{[\tilde{G}(s)G(s)^{-1}(I - \tilde{G}_+(s)F(s)) + \tilde{G}_+(s)F(s)]^{-1}G\ell^*(s)\} \oslash G\ell(s)|$. This is exactly the dynamic GRDG by taking the tuning factor $1/(\epsilon s + 1)$ into account. This term is plotted against ω for different ϵ's, as shown in Figure 5A. As this figure illustrates, for sluggish response (for example, $\epsilon = 100$), generally, static GRDG is enough. In this example, it starts at static GRDG and ends at 1. However, for a tighter performance (for example, small ϵ), the dynamic part of the GRDG become important, as shown in Figure 5A. For "perfect" performance ($\epsilon = 0$), it reduces to d-GRDG. Therefore, the significance of d-GRDG depends a great deal on the performance specification. The frequency range of interest is roughly up to the closed-loop bandwidth ω_b. The results in Figure 5A only amount to part of the closed-loop performance. By taking the integrating term into account, the true performance measure $|S^*|$ is shown in Figure 5B. Actually, S^* in Figure 5B is obtained by multiplying the original term $|\{[\tilde{G}(s)G(s)^{-1}(I - \tilde{G}_+(s)F(s)) + \tilde{G}_+(s)F(s)]^{-1}G\ell^*(s))\} \oslash G\ell(s)|$ (the part of the Figure 5A) with $(\epsilon s)/(\epsilon s + 1)$. Figure 5 clearly shows that the necessity of the d-GRDG depends on the tightness of the desired closed-loop performance.

The d-GRDG is useful in selecting the best controller structure. Consider the following example:

(A)



(B)

Figure 5. Frequency-dependent analysis for Example 2: (a) $|\{[\tilde{G}G^{-1}(I-G_+F)+G_+F]^{-1}G\ell^*\}\circ G\ell|$ and (b) $|S^*|$.

Example 3.

$$G(s)=\begin{bmatrix} \dfrac{-0.8}{(6s+1)} & \dfrac{-0.1}{(6s+1)} & \dfrac{0.1}{(6s+1)} \\ \dfrac{0.1}{(15s+1)^2} & \dfrac{0.1}{(15s+1)^2} & \dfrac{-0.6}{(6s+1)^2} \\ \dfrac{0.1}{(18s+1)^2} & \dfrac{-0.5}{(12s+1)^2} & \dfrac{1.4}{(9s+1)^2} \end{bmatrix}$$

$$\text{and } G\ell(s)=\frac{1}{3s+1}\begin{bmatrix} 0.1 \\ 0.5 \\ 0.6 \end{bmatrix}$$

The RDGA is:

$$\mathcal{B}=\begin{bmatrix} 1.07 & -0.79 & 0.72 \\ -0.03 & 1.89 & -0.86 \\ -0.02 & -0.66 & 1.68 \end{bmatrix}$$

For the first output (y_1), the GRDG (δ_1) is 0.28 ($0.28=1.07+(-0.79)$) for the block diagonal controller (bd[(1,2),3]), which is a much better choice over the full multivariable controller or the diagonal controller. Moreover, the magnitude plot of δ_1, as shown in Figure 6, shows that for the block diagonal controller, $|\delta_1|$ starts to increase from 0.28 at $\omega\cong 0.02$ and approaches to 3 at $\omega\cong 1$. This is exactly the frequency range of interest in the controller design. A controller is designed to have the same closed-loop time constants as the open-loop one, that is, $\tau_{CL}=6$ min. Simulation results, shown in Figure 6, show that the improvement in the performance is not quite good as predicted from the steady-state analysis $\delta_1(0)=0.28$. The ratio of IEs is:

$$\frac{\mathrm{IE}\{G_{bd}\}}{\mathrm{IE}\{G_{\mathrm{full}}\}}=0.28$$

which is exactly what the steady-state analysis shows. However, *d*-GRDG, shown in Figure 7 reveals that the performance is not as good as predicted from steady-state analysis. The ratio of IAEs confirms this.

$$\frac{\mathrm{IAE}\{G_{bd}\}}{\mathrm{IAE}\{G_{\mathrm{full}}\}}=1.07$$

As described earlier, the cancellation in areas occurs even for $\delta_i=0.28$ (>0) in this example.

Note that *d*-GRDG can converge to zero or go to infinity as ω increases. Certainly, this provides a simple way of judging

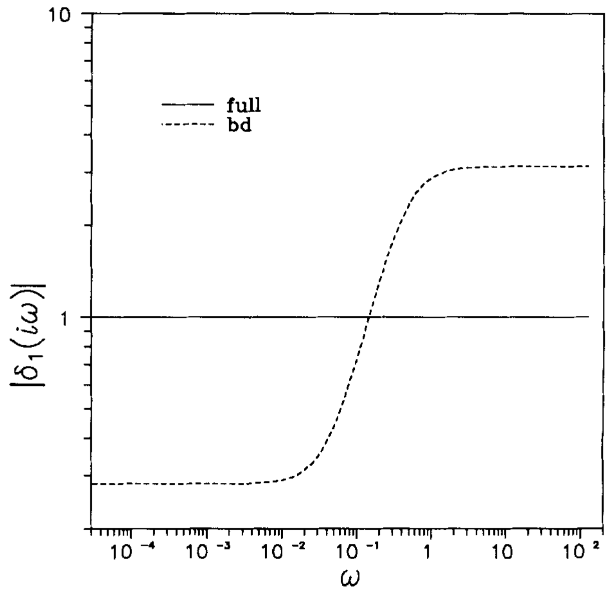


Figure 6. *d*-GRDG ($|\delta_i(i\omega)|$) of Example 3 for full and block diagonal [(1,2),3] controller structures.

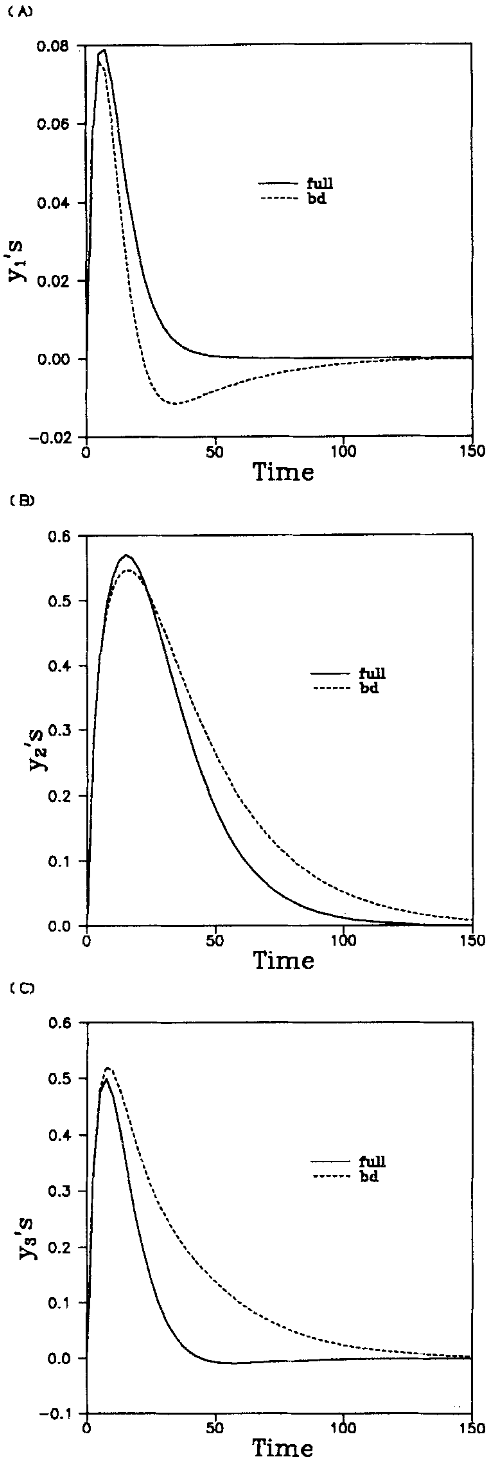


Figure 7. Load responses of example 3 for full and block diagonal [(1,2),3] controller structures.

the disturbance rejection capability dynamically. This is the result of imbalanced pole excesses in $G\ell(s)$. For example, for a system with

$$G(s) = \frac{1}{\tau s + 1}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } G\ell(s) = \frac{1}{\tau_1 s + 1}\begin{bmatrix} \dfrac{k_1}{\tau_2 s + 1} \\ k_2 \end{bmatrix}.$$

Consider the diagonal controller, that is, $\tilde{G} = G_{\text{diag}}$, the d-GRDG is:

$$d\text{-GRDG}(s) = \frac{1}{ad - bc}\begin{bmatrix} ad - \dfrac{abk_2}{k_1}(\tau_2 s + 1) \\ ad - \dfrac{cdk_1}{k_2(\tau_1 s + 1)} \end{bmatrix}$$

It is obvious that as ω goes to infinity, the first element of d-GRDG(s) approaches infinity. Note that the d-GRDG(s) for a full multivariable controller is equal to 1 throughout the frequencies. Therefore, for this example, if a tighter performance is desired dynamically, a better controller is to use both elements for the first output, for example,

$$\Gamma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Dynamic GRDG can be helpful in analyzing the disturbance rejection capability of different controller structures. However, the usefulness of d-GRDG depends on a great deal on the bandwidth (ω_b) of the closed-loop system, as shown in Figures 5 and 6. This is, the frequency range of d-GRDG useful in analyzing the load responses: $\omega < \omega_b$.

Conclusion

As the result of interaction, the controller structure plays an important role in the disturbance rejection capability for a multivariable system. An interaction measure, GRDG, is proposed. It is defined based on IMC structure. GRDG differs from RDG in that all possible controller structures can be analyzed. Furthermore, the physical significance of GRDG is more obvious. GRDG is the ratio of the net load effect (as the result of interaction) over the open-loop load effect (the load effect for a decoupled design). The synthesis of the best controller structure is made easy via the introduction of RDGA. With RDGA (a matrix), the GRDG (a vector) of all possible controller structures can be calculated by simply adding up the elements in the corresponding rows. Qualitative and quantitative analyses of GRDG are given which form the basis for the synthesis of controller structure for disturbance rejection. Servo problem is treated as a special case of disturbance rejection. The relationship between GRDG and RGA (or BRG) is derived for set point responses. Finally, GRDG is extended to a frequency-dependent measure. The usefulness of d-GRDG is discussed. Several examples are used to illustrate the synthesis procedure. The results show that the disturbance rejection capability can be improved via appropriate selection of the controller structure and the synthesis for the optimal controller structure can be done in a straightforward manner using RDGA.

Acknowledgment

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Notation

BRG = block relative gain
bd[(1,2),3] = block diagonal controller with [(1,2),3] structure

bd[(1,3),2] = block diagonal controller with [(1,3),2] structure
bd[1,(2,3)] = block diagonal controller with [1,(2,3)] structure
C = conventional controller
d = load effect
det = determinant
d-GRDG = dynamic GRDG
$\boldsymbol{E}$ = Laplace transformation of $\boldsymbol{e}$
$\mathcal{E}$ = diagonal tuning matrix with entries $\kappa_i \epsilon_i$
$\boldsymbol{e}$ = error vector
$\boldsymbol{F}$ = diagonal filter
$\boldsymbol{G}$ = process transfer function matrix
$\tilde{\boldsymbol{G}}$ = process model in IMC for defining the controller structure
$\boldsymbol{Gc}$ = IMC controller
$\boldsymbol{G\ell}$ = load transfer function vector
g_{ij} = ijth element of matrix $\boldsymbol{G}$
$\hat{g}_{ij}$ = ijth element of matrix $\boldsymbol{G}^{-1}$
$g\ell_i$ = ith element of vector $\boldsymbol{G\ell}$
GRDG = generalized relative disturbance gain
IE = integrated error
IAE = integrated absolute error
L = load variable
$\boldsymbol{P}$ = permutation matrix
RDG = relative disturbance gain
RDGA = relative disturbance gain array
RGA = relative gain array
RHP = right half-plane
s = Laplace transformation variable
S^* = performance measure defined as $e_{CL}(s) \oslash e_{OL}(s)$
S_I = diagonal input scaling matrix
S_0 = diagonal output scaling matrix
t = time variable
$\boldsymbol{u}$ = input variable
vdm = transformation of a vector into a matrix by placing element of the vector on the corresponding diagonal of the matrix
$\boldsymbol{y}$ = output variable
z_i = ith column vector of identity matrix

Greek letters

β_i = ith element of RDG proposed by Stanley et al. (1985)
β_{ij} = ijth element of RDGA
δ_i = ith element of GRDG
ϵ_i = ith tuning constant in IMC filter
Γ = structure selection matrix as defined by Eq. 13
γ_{ij} = ijth element of structure matrix
κ_i = positive integer
ω = frequency

Subscripts

b = bandwidth
bd = block diagonal structure
CL = closed-loop
diag = diagonal structure
OL = open-loop
p = permutation
tri = triangular structure

Superscripts

set = set point
T = transpose
-1 = inversion
* = modified case

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Appendix A

If a vector $\boldsymbol{G\ell}$ is premultiplied with a permutation matrix $\boldsymbol{P}$, then $\mathrm{vdm}(\boldsymbol{P}\cdot\boldsymbol{G\ell}) = \boldsymbol{P}\cdot\mathrm{vdm}(\boldsymbol{G\ell})\cdot\boldsymbol{P}^T$.

Proof: Assume that the ith and jth elements are permuted, that is $(g\ell_i)_P = g\ell_j$ and $(g\ell_j)_P = g\ell_i$. Then, the diagonal matrix $\mathrm{vdm}(\boldsymbol{G\ell})$ should be premultiplied with $\boldsymbol{P}$ for the exchange of the ith row with the jth row and post-multiplied with $\boldsymbol{P}^T$ for the exchange of the ith column with the jth column. Therefore, $\boldsymbol{P}\cdot\mathrm{vdm}(\boldsymbol{G\ell})\cdot\boldsymbol{P}^T$ is back to be a diagonal matrix with these two elements permuted. We have:

$$\mathrm{vdm}(\boldsymbol{P}\cdot\boldsymbol{G\ell}) = \boldsymbol{P}\cdot\mathrm{vdm}(\boldsymbol{G\ell})\cdot\boldsymbol{P}^T. \tag{A1}$$

Appendix B

Proof of Eq. 29. The structure of $\tilde{\boldsymbol{G}}$ can be expressed as

$$\tilde{\boldsymbol{G}} = \boldsymbol{G} \otimes \Gamma = \begin{bmatrix} g_{11}\gamma_{11} & g_{12}\gamma_{12} & \cdots g_{1n}\gamma_{1n} \\ g_{21}\gamma_{21} & g_{22}\gamma_{22} & \cdots g_{2n}\gamma_{2n} \\ \vdots & \cdots & \cdots \\ g_{n1}\gamma_{n1} & g_{n2}\gamma_{n2} & \cdots g_{nn}\gamma_{nn} \end{bmatrix} \tag{B1}$$

Decompose $\tilde{G}$ columnwise

$$\tilde{G}=\begin{bmatrix} g_{11}\gamma_{11} & 0 & \cdots 0 \\ g_{21}\gamma_{21} & 0 & \cdots 0 \\ \cdots & \cdot & \cdots \\ g_{n1}\gamma_{n1} & 0 & \cdots 0 \end{bmatrix}+\begin{bmatrix} 0 & g_{12}\gamma_{12} & 0 & \cdots 0 \\ 0 & g_{22}\gamma_{22} & 0 & \cdots 0 \\ & \cdots & \cdot & \cdots \\ 0 & g_{n2}\gamma_{n2} & 0 & \cdots 0 \end{bmatrix}+\cdots+\begin{bmatrix} 0 & 0 & \cdots & 0 & g_{1n}\gamma_{1n} \\ 0 & 0 & \cdots & 0 & g_{2n}\gamma_{2n} \\ & & \cdots & & \cdots \\ 0 & 0 & \cdots & 0 & g_{nn}\gamma_{nn} \end{bmatrix} \quad \text{(B2)}$$

From the definition (Eq. 20), the GRDG vector becomes:

$$\mathrm{GRDG}=(\tilde{G}\cdot G^{-1}G\ell)\oslash G\ell=\begin{bmatrix} \delta_{11}\gamma_{11} \\ \delta_{21}\gamma_{21} \\ \vdots \\ \delta_{n1}\gamma_{n1} \end{bmatrix}+\begin{bmatrix} \delta_{12}\gamma_{12} \\ \delta_{22}\gamma_{22} \\ \vdots \\ \delta_{n2}\gamma_{n2} \end{bmatrix}+\cdots+\begin{bmatrix} \delta_{1n}\gamma_{1n} \\ \delta_{2n}\gamma_{2n} \\ \vdots \\ \delta_{nn}\gamma_{nn} \end{bmatrix}=\begin{bmatrix} \sum_{j=1}^{n}\delta_{1j}\gamma_{1j} \\ \sum_{j=1}^{n}\delta_{2j}\gamma_{2j} \\ \vdots \\ \sum_{j=1}^{n}\delta_{nj}\gamma_{nj} \end{bmatrix} \quad \text{(B3)}$$

Appendix C

For a step change in L, the integrated error (IE) under the IMC structure (Figure 2) is:

$$\mathrm{IE}\{G\}=\int_0^{\infty} e(t)dt=\lim_{s\to 0} e(s)$$
$$=\lim_{s\to 0} -(I-\tilde{G}\cdot Gc)(I+(G-\tilde{G})Gc)^{-1}G\ell/s \quad \text{(C1)}$$

Let $Gc=\tilde{G}_-^{-1}F$ and F be the form as Eq. 9, that is,

$$F=\mathrm{diag}\left(\frac{1}{(\tilde{\epsilon}_i s+1)^{\tilde{\kappa}_i}}\right) \quad (9)$$

where $\tilde{\kappa}_i$ is some integer such that Gc is proper.

$$\mathrm{IE}\{G\}=\lim_{s\to 0} -(I-F)(I+G\cdot\tilde{G}_-^{-1}F-F)^{-1}G\ell/s$$
$$=\lim_{s\to 0} -(sI+G\cdot\tilde{G}_-^{-1}\cdot s\cdot F(I-F)^{-1})^{-1}G\ell \quad \text{(C2)}$$

The term $s\cdot F(I-F)^{-1}$ is calculated first. Let us take the limit for the ith element of it. Using L'Hospital rule, we have:

$$\lim_{s\to 0}\frac{s}{(\tilde{\epsilon}_i s+1)^{\tilde{\kappa}_i}}\cdot\frac{(\tilde{\epsilon}_i s+1)^{\tilde{\kappa}_i}}{(\tilde{\epsilon}_i s+1)^{\tilde{\kappa}_i}-1}=\lim_{s\to 0}\frac{s}{(\tilde{\epsilon}_i s+1)^{\tilde{\kappa}_i}-1}$$
$$=\lim_{s\to 0}\frac{1}{\tilde{\kappa}_i(\tilde{\epsilon}_i s+1)^{\tilde{\kappa}_i}\tilde{\epsilon}_i}=\frac{1}{\tilde{\kappa}_i\tilde{\epsilon}_i} \quad \text{(C3)}$$

Therefore, Eq. C2 can be obtained as:

$$\mathrm{IE}\{\tilde{G}\}=-\tilde{\mathcal{E}}\cdot\tilde{G}(0)G(0)^{-1}G\ell(0) \quad \text{(C4)}$$

where $\tilde{\mathcal{E}}=\mathrm{diag}(\tilde{\kappa}_i\tilde{\epsilon}_i)$. For the controller with the full structure, that is, $\tilde{G}=G$, we have:

$$\mathrm{IE}\{G\}=-\mathcal{E}\cdot G\ell(0) \quad \text{(C5)}$$

where $\mathcal{E}=\mathrm{diag}(\kappa_i\epsilon_i)$. Note that $\tilde{\mathcal{E}}$ and $\mathcal{E}$ are exactly the tuning matrices for the controllers with different structures ($\tilde{G}$ and G). Therefore, under this circumstance the ratio of IEs becomes:

$$\mathrm{IE}\{\tilde{G}\}\oslash\mathrm{IE}\{G\}=(\tilde{\mathcal{E}}\cdot\tilde{G}\cdot G^{-1}\cdot G\ell)\oslash(\mathcal{E}\cdot G\ell)$$
$$=(\tilde{\mathcal{E}}\oslash\mathcal{E})\cdot\mathrm{GRDG} \quad \text{(C6)}$$

Appendix D

Consider the IMC control system, as shown in Figure 2, the controller Gc is designed according to Eqs. 6–8. $G\ell$ is the open-loop load transfer function between y and L. $G\ell_{CL}$ is the closed-loop load transfer function vector when y is under integral control. For the ith element of $G\ell$, assume that $g\ell_i>0$. If (a) G, $\tilde{G}$, and $G\ell$ are strictly proper and (b) $\delta_i<0$, then y_i crosses the set point at least one time.

Proof. From Eq. 5, the closed-loop load function $G\ell_{CL}$ can be expressed as:

$$G\ell_{CL}=(I-\tilde{G}(s)\cdot Gc(s))[I+(G(s)-\tilde{G}(s))Gc(s)]^{-1}G\ell(s) \quad \text{(D1)}$$

Considering the initial slope of y, we have

$$\lim_{t\to 0}\frac{\partial y}{\partial t}=\lim_{s\to\infty} s^2(I-\tilde{G}(s)\cdot Gc(s))[I+(G(s)-\tilde{G}s))Gc(s)]^{-1}G\ell(s)/s \quad \text{(D2)}$$

Since Gc is proper and with assumption of condition (b), Eq. D2 becomes

$$\lim_{t\to 0}\frac{\partial y}{\partial t}=\lim_{s\to\infty} s\cdot G\ell(s) \quad \text{(D3)}$$

For the ith output with $g\ell_i>0$, we have

$$\lim_{t\to 0}\frac{\partial y_i}{\partial t}=\lim_{s\to\infty} s\cdot g\ell_i>0 \quad \text{(D4)}$$

From Eq. D4, we know y_i has positive initial slope. Again from Appendix C, for $\delta_i<0$, we have larger net area below the set point. This implies that the response curve goes up initially, then crosses the set point at least once and ends up with a larger net area below the set point, that is |Area| above the set point < |Area| below the set point.

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